Note on Continuous Markov Chain

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1 Semi-Markov Process

Definition 1.1 (Semi-Markov Process)

A semi-Markov process is one that changes states in accordance with a Markov chain but takes a random amount of time between changes. More specifically consider a stochastic process with states 0,1,..., which is such that, whenever it enters state $i, i \geq 0$:

1. *The next state it will enter is state j with probability $P_{ij}, i, j \geq 0$.*
2. *Given that the next state to be entered is state j, the time until the transition from i to j occurs has distribution F_{ij}.*

If we let $Z(t)$ denote the state at time t, then $\{Z(t), t \geq 0\}$ is called a semi-Markov process.

Remark A Markov chain is a semi-Markov process in which

$$F_{ij}(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$$

Note that semi-Markov process may not possess Markovin Property.

Definition 1.2 (Embedded Markov Chain)

If we let X_n denote the nth state visited, then $\{X_n, n \geq 0\}$ is a Markov chain with transition probabilities P_{ij}. It is called the embedded Markov chain of the semi-Markov process.

Definition 1.3 (Irreducible Semi-Markov process)

The semi-Markov process is irreducible if the embedded Markov chain is irreducible as well.

Definition 1.4 (Time spends in state i per transition)

H_i denote the distribution of time that the semi-Markov process spends in state i before making a transition.

$$H_i(t) = \sum_j P_{ij} F_{ij}(t)$$

Here we use μ_i denote its mean

$$\mu_i = \int_0^\infty x dH_i(x)$$

Definition 1.5 (Time between state i)

T_{ii} denote the time between successive transitions into state i and let $\mu_{ii} = E[T_{ii}]$.

Proposition 1.1 (Infinite's state probability)

If the semi-Markov process is irreducible and if T_{ii} has a nonlattice distribution with finite mean, then

$$P_i \equiv \lim_{t\to\infty} P\{Z(t) = i \mid Z(0) = j\}$$

exists and is independent of the initial state. Furthermore, $P_i = \frac{\mu_i}{\mu_{ii}}$

Corollary 1.1 (Long-run proportion of time in state i)

If the semi-markov process is irreducible and $\mu_{ii} < \infty$, then with probability 1,

$$\frac{\mu_i}{\mu_{ii}} = \lim_{t\to\infty} \frac{\text{amount of time in } i \text{ during } [0,t]}{t}$$

Theorem 1.1 (Long-run probability in state i: P_i)

Suppose the condition of Proposition 1.1 and suppose further that the embedded Markov chain $\{X_n, n \geq 0\}$ is positive recurrent. Then

$$P_i = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$$

where π_j means the proportion of X_n that equal j, μ_j is the mean time spent in state j per transition, and P_i is actually the long-run proportion of time it is in state i.

2 Continuous-Time Markov Chains

Definition 2.1 (Continuous-Time Markov Chains)

Consider a continuous-time stochastic process $\{X(t), t \geq 0\}$ taking on values in the set of non-negative integers. In analogy with the definition of a discrete-time Markov chain, we say that the process $\{X(t), t \geq 0\}$ is a continuous-time Markov chain if for all $s, t \geq 0$, and nonnegative integers $i, j, x(u), 0 \leq u \leq s$,

$$P\{X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s\} = P\{X(t+s) = j \mid X(s) = i\} \quad \text{(Markovian property)}$$

Lemma 2.1 (Independence of time and next state)

The amount of time the process spends in state i and the next state visited is independent.

Definition 2.2 (Time stay in i per transition)

τ_i denote the amount of time that the process stays in state i before making a transition into a different state.

Lemma 2.2 (Exponential τ_i)

Random variable τ_i is memoryless and must be exponentially distributed.

Proof

$$
\begin{aligned}
P\{X(u)=i, s<u\leq t+s \mid X(u)=i, 0\leq u\leq s\} &= P\{X(u)=i, s<u\leq t+s \mid X(s)=i\} \\
&= P\{X(u)=i, 0<u\leq t \mid X(0)=i\}
\end{aligned}
$$

$$\Downarrow$$

$$P\{\tau_i > s+t \mid \tau_i > s\} = P\{\tau_i > t\}$$

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Definition 2.3 (Continuous-Time Markov Chains 2)

Continuous-Time Markov Chain is a stochastic process having the properties that each time it enters state i:

1. *the amount of time it spends in state i before making a transition into a different state, is exponentially distributed with rate v_i, and*
2. *when the process leaves state i, it will next enter state j with some probability P_{ij}, where $\sum_{j\neq i} P_{ij} = 1$ ($P_{ii}=0$).*

Remark By exponential distribution, a continuous-time Markov chain is a semi-markov process where the expected time spent in state i is $1/v_i$.

Definition 2.4 (Regular Chain)

A continuous-time Markov chain with property $0\leq v_i<\infty \quad \forall i$ is said to be regular.

Definition 2.5 (Absorbing state)

If $v_i=0$, then state i is called absorbing since once entered it is never left.

Definition 2.6 (Transition Rate from i to j: q_{ij})

q_{ij} is the rate when in state i that the process makes a transition into state j, we call q_{ij} the transition rate from i to j.

$$q_{ij} = v_i P_{ij}, \quad \text{for all } i\neq j \cdot \left(\frac{\text{Imply}}{\sum_{j\neq i} P_{ij}=1} \rightarrow \sum_{j\neq i} q_{ij} = v_i\right)$$

Here when the process is in state j it leaves at rate v_j, and P_{ij} is the probability that it then goes to j.

Definition 2.7 (Continuous-Time Markov Chains 3)

Consider independent exponential random variables X_{ij} with rate q_{ij} to be associated with the possible transition from i to j. When the process enters a given state i, the next transition occurs after $\min\{X_{ij}: j \neq i\}$ *time units. The probability for j to be the next state is* $P\{X_{ij} = \min\{X_{i,j'}: j' \neq i\}\}$.

$$\min\{X_{ij}: j \neq i\} \sim E(\sum_{j \neq i} q_{ij} = v_i)$$

$$P\{X_{ij} = \min\{X_{i,j'}: j' \neq i\}\} = q_{ij} / \sum_{j' \neq i} q_{i,j'} = P_{ij} v_i / v_i = P_{ij}$$

Definition 2.8 (Probability from i to j after t: $P_{ij}(t)$)

$P_{ij}(t)$ is the probability that a Markov chain, presently in state i, will be in state j after an additional time t.

$$P_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\}$$

3 Birth and Death Process

Definition 3.1 (Birth and Death process)

A continuous-time Markov chain with states $0, 1, \ldots$ for which $q_{ij} = 04$ whenever $j \notin \{i-1, i+1\}$ is called a birth and death process. The values $\{\lambda_i, i \geq 0\}$ and $\{\mu_i, i \geq 1\}$ are called respectively the birth rates and the death rates.

$$\lambda_i = q_{i,i+1}, \mu_i = q_{i,i-1} \quad \underrightarrow{\sum_j q_{ij} = v_i} \quad v_i = \lambda_i + \mu_i, P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}$$

Say that $X_i \sim E(\lambda_i)$ denotes the time until the next birth whenever there are i person, and $Y_i \sim E(\mu_i)$ denotes the time until the next death, then $\tau_i = \min\{X_i, Y_i\} \sim E(\lambda_i + \mu_i = v_i)$ is the time until the next transition.

$$\text{the event } \{\tau_i = X_i\} \text{ (resp., } \{\tau_i = Y_i\}) \Longleftrightarrow \text{the next transition is a birth (resp., a death)}$$

$$P\{\tau_i = X_i\} = \frac{\lambda_i}{\lambda_i + \mu_i} = P_{i,i+1} \quad \text{and} \quad P\{\tau_i = Y_i\} = \frac{\mu_i}{\lambda_i + \mu_i} = P_{i,i-1}$$

Lemma 3.1 (Forward Equations for Birth and Death Process)

$$P'_{i0}(t) = \sum_{k \neq 0} q_{k0} P_{ik}(t) - v_0 P_{i0}(t) = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t), \quad j \neq 0$$

Lemma 3.2 (P_i for Birth and Death Process)

State	*Rate Process Leaves*		*Rate Process Enters*
0	$\lambda_0 P_0$	$=$	$\mu_1 P_1$
$n, n > 0$	$(\lambda_n + \mu_n) P_n$	$=$	$\mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$

$$\Downarrow$$

$$\begin{aligned}
&\lambda_0 P_0 = \mu_1 P_1 \\
&\lambda_1 P_1 = \mu_2 P_2 + (\lambda_0 P_0 - \mu_1 P_1) = \mu_2 P_2 \\
&\lambda_2 P_2 = \mu_3 P_3 + (\lambda_1 P_1 - \mu_2 P_2) = \mu_3 P_3 \\
&\lambda_n P_n = \mu_{n+1} P_{n+1} + (\lambda_{n-1} P_{n-1} - \mu_n P_n) = \mu_{n+1} P_{n+1}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
P_1 &= \frac{\lambda_0}{\mu_1} P_0 \\
P_2 &= \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 \\
P_3 &= \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \\
P_n &= \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} P_0
\end{aligned}$$

$$\Downarrow \quad 1 = P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1}$$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \right]^{-1} \qquad P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n \left(1 + \sum_{n'=1}^{\infty} \frac{\lambda_{n'-1} \lambda_{n'-2} \cdots \lambda_1 \lambda_0}{\mu_{n'} \mu_{n'-1} \cdots \mu_2 \mu_1} \right)}, \quad n \geq 1$$

The condition for the limiting distribution to exist is $\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} < \infty$.

Definition 3.2 (Pure Birth process)

A birth and death process is said to be a pure birth process if $\mu_n = 0$ *for all* n.

Remark For example, poisson process is a pure birth process with constant birth rate $\lambda_n = \lambda$.

Lemma 3.3 (Property for Pure Birth process)

The forward equations are

$$\begin{aligned}
P'_{ii}(t) &= -\lambda_i P_{ii}(t) \\
P'_{ij}(t) &= \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t), \quad j > i.
\end{aligned}$$

By $P_{ii}(0) = 1$ *and solving the first equation, we have* $P_{ii}(t) = e^{-\lambda_i t}$*, and it means the probability that the time until a transition from state* i *is greater than* t.

By $P_{ij}(0) = 0$ *and solving the second equation, we have* $P_{ij}(t) =$

$e^{-\lambda_j t}\int_0^t e^{\lambda_j s}\lambda_{j-1}P_{i,j-1}(s)ds, \quad j > i.$

$$e^{\lambda_j t}\lambda_{j-1}P_{i,j-1}(t) = e^{\lambda_j t}\left[P'_{ij}(t) + \lambda_j P_{ij}(t)\right] = \frac{d}{dt}\left[e^{\lambda_j t}P_{ij}(t)\right]$$

Follow this track, we can derive $P_{ij}(t) = \frac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!}$ for Poisson process by induction method.

Proposition 3.1

An ergodic birth and death process is in steady state time reversible.

Definition 3.3 (Birth and Death Process with new members)

Lemma 3.4 ($E[X(t) \mid X(0) = i]$)

4 Kolmogorov Differential Equations

Lemma 4.1 (Limiting $P_{ij}(t)$)

1. $P_{ii}(t) = 1 - v_i t + o(t)$, *i.e.*, $\lim_{t\to 0}\frac{1-P_{ii}(t)}{t} = v_i$.
2. $P_{ij}(t) = q_{ij}t + o(t)$, *i.e*, $\lim_{t\to 0}\frac{P_{ij}(t)}{t} = q_{ij}, i \neq j$.

Lemma 4.2 (Chapman-Kolmogorov Equations)

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s) \quad \forall s, t$$

Theorem 4.1 (Kolmogorov's Backward Equations)

$$P'_{ij}(t) = \sum_{k\neq i} q_{ik}P_{kj}(t) - v_i P_{ij}(t) = \sum_{k\neq i} q_{ik}\left(P_{kj}(t) - P_{ij}(t)\right) \quad \forall i, j, t \geq 0$$

Proof

$$\begin{cases} P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(h)P_{kj}(t) - P_{ij}(t) = \sum_{k\neq i} P_{ik}(h)P_{kj}(t) - \left[1 - P_{ii}(h)\right]P_{ij}(t) \\ \lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h} = \lim_{h\to 0}\left\{\sum_{k\neq i}\frac{P_{ik}(h)}{h}P_{kj}(t) - \frac{1-P_{ii}(h)}{h}P_{ij}(t)\right\} \end{cases}$$

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Remark Here q_{ik} is the rate moving from i to k in an incremental interval. $P_{kj}(t) - P_{ij}(t)$ means the probability the system changes to state j over length of time t if $i \to k$ happens, i.e., q_{ik}. And $P'_{ij}(t)$ means the incremental change in $P_{ij}(t)$.

Theorem 4.2 (Kolmogorov's Forward Equations)

$$P'_{ij}(t) = \sum_{k\neq i} q_{kj}P_{ik}(t) - v_j P_{ij}(t) \quad \textit{Suitable regularity conditions}$$

Remark The incremental change in $P_{ij}(t)$ is equal to the difference in two terms. One is the sum over k of the probability of being in state k at time t and then moving to j in the final incremental interval, the other is the probability of bing in state j at time t and then moving away in the final incremental interval.

Proof

$$\begin{cases} P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(t)P_{kj}(h) - P_{ij}(t) = \sum_{k\neq j} P_{ik}(t)P_{kj}(h) - \left[1 - P_{jj}(h)\right] P_{ij}(t) \\ \lim_{h\to 0} \frac{P_{ij}(t+h)-P_{ij}(t)}{h} = \lim_{h\to 0} \left\{ \sum_{k\neq j} P_{ik}(t)\frac{P_{kj}(h)}{h} - \frac{1-P_{jj}(h)}{h} P_{ij}(t) \right\} \end{cases}$$

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Theorem 4.3 (Balance Equation)

$$v_j P_j = \sum_i P_i q_{ij} \quad \textit{and} \quad \sum_j P_j = 1$$

Here $v_j P_j$ means the rate at which the process leaves state j, $\sum_i P_i q_{ij}$ means the rate at which the process enters state j.

Remark It means in any interval $(0,t)$, the number of transitions into state j must be equal to the number of transitions out of state j.

$$(\#\text{ transitions into } j \text{ in } (0,t]) - (\#\text{ transitions out of } j \text{ in } (0,t]) = \begin{cases} 1 & X(0)\neq j, X(t) = j \\ -1 & X(0) = j, X(t) \neq j \\ 0 & X(0) = j, X(t) = j \\ 0 & X(0) \neq j, X(t) \neq j \end{cases}$$

Proof

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1$$

$$\Downarrow P_j = \frac{\pi_j/v_j}{C} (P_j = \frac{\pi_j/v_j}{\sum_i \pi_i/v_i}) \quad \sum_j P_j = 1$$

$$v_j P_j = \sum_i v_i P_i P_{ij} \quad \text{and} \quad \sum_j P_j = 1$$

$$\Downarrow q_{ij} = v_i P_{ij}$$

$$v_j P_j = \sum_i P_i q_{ij} \quad \text{and} \quad \sum_j P_j = 1$$

$$\Uparrow$$

$$0 = \sum_{k \neq j} q_{kj} P_k - v_j P_j$$

$$\Uparrow P_{ij}(t) \text{ is bounded } (0 \leq P_{ij}(t) \leq 1),\ P'_{ij}(t) \text{ converges to } 0$$

$$\lim_{t\to\infty} P'_{ij}(t) = \lim_{t\to\infty} \left[\sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \right] = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

$$\Uparrow$$

$$P'_{ij}(t) = \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \quad \text{Forward Equations}$$

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5 Reversed Chain and Time reversibility

Definition 5.1 (Ergodic chain)

1. *When a continuous-time Markov chain is irreducible and the limiting probabilities $P_j > 0$ for all j, we say that the chain is ergodic.*
2. *When the embedded discrete-time Markov chain is irreducible and positive recurrent, we say that the chain is ergodic.*

Definition 5.2 (Steady state)

Steady state means an ergodic continuous-time Markov chain has been in operation an infinitely long time, that is,

$$P\{X(t) = j\} = P_j$$

Definition 5.3 (Reversed chain)

Consider a continuous-time Markov chain in steady state going backwards in time, the reverse process is also a continuous-time Markov chain.

$$P\{X(t-s) = j \mid X(t) = i, X(y), y > t\} = P\{X(t-s) = j \mid X(t) = i\}$$

Lemma 5.1 (Reversed chain's property)

1. *The embedded discrete-time Markov chain transition probabilities* P^*_{ij} *is given by* $P^*_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$.
2. *The amount of time spent in a state is the same regardless of forward or backward, that is,* v_i *is the same (the time spent in* i *follows* $Exp(v_i)$*).*
3. *Transition Rate:* $q^*_{ij} = v_i P^*_{ij} = \frac{v_j P_j P_{ji}}{P_i} = \frac{P_j q_{ji}}{P_i}$*, that is,* $P_i q^*_{ij} = P_j q_{ji}$*. Here* $P_i q^*_{ij}$ *is the rate at which the reverse chain makes a transition from* i *to* j*,* $P_j q_{ji}$ *is the rate at which the forward chain makes a transition from* j *to* i*.*

Proof (3) holds because $P_k = \frac{\pi_k/v_k}{C}$, where $C = \sum_i \pi_i/v_i$, then $\frac{\pi_j}{\pi_i} = \frac{v_j P_j}{v_i P_i}$. ■

Definition 5.4 (Time reversible)

The stationary continuous-time Markov chain is said to be time reversible if the reverse process follows the same probabilistic law as the original process. That is, it is time reversible if for all i *and* j

$$q^*_{ij} = q_{ij}$$

which is equivalent to

$$P_i q_{ij} = P_j q_{ji} \quad \forall i, j$$

That is, the condition of time reversibility is that the rate at which the process goes directly from state i *to state* j *is equal to the rate at which it goes directly from* j *to* i*.*

Lemma 5.2

Let q_{ij} *denote the transition rates of an irreducible continuous-time Markov chain. If we can find a collection of numbers* $q^*_{ij}, i, j \geq 0, i \neq j$*, and a collection of nonnegative numbers* $P_i, i \geq 0$*, summing to unity, such that*

$$P_i q_{ij} = P_j q^*_{ji}, \quad i \neq j \quad \text{and} \quad \sum_{j \neq i} q_{ij} = \sum_{j \neq i} q^*_{ij}, \quad i \geq 0$$

then q^*_{ij} *are the transition rates for the reversed chain and* P_i *are the limiting probabilities (for both chains).*

Lemma 5.3 (Trancated Chain)

A time-reversible chain with limiting probabilities $P_j, j \in S$*, that is truncated to the set* $A \subset S$ *and remains irreducible is also time reversible and has limiting probabilities*

$$P^A_j = \frac{P_j}{\sum_{k \in A} P_k}, \quad j \in A$$